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Infinite sequence of Poincaré group extensions: structure and dynamics

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Abstract
We study the structure and dynamics of the infinite sequence of extensions of the Poincaré algebra whose method of construction was described in a previous paper (Bonanos and Gomis *J. Phys. A: Math. Theor.* **42** (2009) 145206 (arXiv:hep-th/0808.2243)). We give explicitly the Maurer–Cartan (MC) 1-forms of the extended Lie algebras up to level 3. Using these forms and introducing a corresponding set of new dynamical couplings, we construct an invariant Lagrangian, which describes the dynamics of a distribution of charged particles in an external electromagnetic field. At each extension, the distribution is approximated by a set of moments about the world line of its center of mass and the field by its Taylor series expansion about the same line. The equations of motion after the second extensions contain back-reaction terms of the moments on the world line.

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1. Introduction

In a recent paper [1], we have studied aspects of the Chevalley–Eilenberg cohomology of the Galilei and Poincaré groups. In particular, we have seen that, at degree two, there is an infinite sequence of Lie algebra extensions, beginning with the Galilei or Poincaré algebras. We recall that the extensions found are non-central since the corresponding generators transform non-trivially under the corresponding normal subalgebra of the unextended algebra.

In this paper, we study further the infinite sequence of extensions of the Poincaré algebra. We study the tensor structure of the extensions and their physical interpretations. It is known that the Poincaré group in $d+1$ ($d > 1$) has no central extensions[4] and that it has a non-central antisymmetric tensor extension [8]. Physically, this extension corresponds to the symmetries of a relativistic particle in a constant electromagnetic field and is known as the Maxwell group [9, 10]. A modification of the Poincaré algebra having only two Lorentz generators, that leaves a constant background electromagnetic field invariant, allows an extension with two central charges that can be interpreted as the electric and magnetic charge (BCR algebra [11]). Non-central extensions have also been considered for the diffeomorphism gauge algebra, see for example [12].

Both central and non-central extensions are controlled by Chevalley–Eilenberg cohomology theory, see for example [13]. Here we compute in a systematic, almost algorithmic[5], way the most general CE cohomology groups at form degree two. As we will see, the non-trivial forms belong to different representations of the subgroup of Lorentz transformations of the Poincaré group, in general with mixed symmetries. The Lorentz transformations are a subgroup of the automorphism group and constitute a normal subgroup of each extended group. We can associate with any non-central extension a Young tableau.

As discussed in [1], the first non-central extension of the Poincaré algebra is obtained by calculating all possible non-trivial closed 2-forms of the subgroup of space–time translations. The forms are closed with respect to the exterior differential operator d. The complete extended algebra is constructed from the original algebra and the extensions by incorporating their transformation properties under Lorentz transformations, which is equivalent to replacing the exterior differential operator d by the corresponding 'covariant' operator $\mathrm{d}+M\wedge$, M being the zero-curvature connection associated with the Lorentz generators: $\mathrm{d}M + M \wedge M = 0$.

Once we have an extended algebra, we can further extend it by applying the same procedure to an extended set of 'translations' which includes all generators except those of the Lorentz subgroup. In this way, we obtain new extensions whose generators belong to higher dimensional representations of the Lorentz group. This procedure does not end resulting in an infinite sequence of groups—extensions of the Poincaré group. In the limit, our procedure formally defines an infinite Lie algebra. However, we cannot prove this result.

We do not have a precise mathematical interpretation for this infinite Lie algebra. We find it intriguing, however, that the generator content of this algebra is organized in levels like the Lorentzian Kac–Moody algebras that are conjectured to be a symmetry of supergravity; see, for example, [15] for the E_{11} approach and [16] for the E_{10} approach.

In order to obtain a possible physical interpretation of this infinite sequence of extensions of the Poincaré group, we construct a relativistic particle Lagrangian, invariant under the extended algebra, by using the MC forms. We also introduce tensor coupling 'constants' that we consider as new dynamical variables. These tensor couplings are invariant under the extended symmetries.

The form of the equations of motion following from this Lagrangian leads us to the conclusion that the physical system in question is a distribution of charged particles, described collectively as a particle with a set of multipole moments, moving in a fixed background electromagnetic field. The multipoles can be considered as Goldstone bosons. The background field is described by its Taylor series expansion about the world line of the 'particle', higher

[4] In 1 + 1 dimensions, there is one central extension [2], which has been used to study several problems of gravity and Moyal quantization, see for example [3–5]. Also, in spaces R^{2n} and R^{4n} with automorphism groups $U(n)$ and $Sp(n) \times SU(2)$ (Kähler and hyper-Kähler geometries), Galperin *et al* [6, 7] have obtained complex first-level central extensions (a triplet in hyper-Kähler).

[5] The calculations make use of the first author's Mathematica package EDC (exterior differential calculus) [14]. The procedure is described in detail in [1].

terms in the series (and higher moments) appearing with every extension. Moreover, new terms in the equation of motion of the 'particle' due to back-reaction terms involving the moments appear. These results are obtained by integrating the equations of motion for the coupling fields and plugging the solutions into the equations of motion of the particle coordinates. Once we choose a particular solution, the equations of motion for the particle coordinates imply a spontaneous breaking of the symmetries of the extended algebra.

The organization of the paper is as follows. In section 2, we introduce our notation and conventions and obtain the first-level extensions. We then find explicit expressions for the generators of the extended group, construct the Lagrangian and deduce the transformations of the fields that leave the Lagrangian invariant. Finally, we obtain the equations of motion for all dynamical variables. In sections 3 and 4, we repeat these steps for the second and third extensions. We also give the defining equations for the fourth-level extensions. In section 5, we point out that it is advantageous to consider the Young Tableau symmetries of the different extensions, and how symmetry considerations can determine the structure of higher extensions. Finally, in section 6, we compare our results with other approaches for constructing theories with higher symmetry and discuss the implications.

2. The Poincaré group in 3+1 dimensions

The generators of the unextended Poincaré algebra are the translations P_a and the Lorentz transformations M_{ab}, where the tensor indices take the values (0, 1, 2, 3). Denoting by η_{ab} the Minkowski metric, the algebra is given by[6]

$$\begin{aligned} [M_{ab}, M_{cd}] &= -\mathrm{i}(\eta_{bc}M_{ad} - \eta_{bd}M_{ac} + \eta_{ad}M_{bc} - \eta_{ac}M_{bd}), \\ [P_a, M_{bc}] &= -\mathrm{i}(\eta_{ab}P_c - \eta_{ac}P_b). \end{aligned} \tag{2.1}$$

As described in [1], in order to construct the extensions we make use of the left invariant Maurer–Cartan (MC) form, defined by

$$\Omega = -\mathrm{i}g^{-1}\,\mathrm{d}g, \tag{2.2}$$

where g represents a general element of the Poincaré group. The MC form satisfies the Maurer–Cartan equation

$$\mathrm{d}\Omega + \mathrm{i}\Omega\wedge\Omega = 0. \tag{2.3}$$

In components, the MC 1-form is written, for a generic Lie algebra, as

$$\Omega = X_A\mathcal{X}^A, \tag{2.4}$$

where X_A are the generators of the Lie algebra satisfying

$$[X_B, X_C] = \mathrm{i}f^A{}_{BC}X_A \tag{2.5}$$

and $\mathcal{X}^A$ are corresponding 1-forms[7]. Throughout this paper, we use the same capital letters in plain and calligraphic font to denote generators and associated 1-forms. The MC equation (2.3) implies that the 1-forms $\mathcal{X}^A$ satisfy

$$\mathrm{d}\mathcal{X}^A = \tfrac{1}{2}f^A{}_{BC}\mathcal{X}^B\wedge\mathcal{X}^C. \tag{2.6}$$

[6] Although the algebra is real and the imaginary units can be made to disappear by replacing all generators G by $\mathrm{i}G'$, we prefer to leave the i's in the equations because then we can interpret the generators as Hermitian operators.

[7] In general, the generator indices will refer to multiple-index tensors with symmetries. When such indices are summed as in (2.4), (2.5), additional numerical factors must be introduced to compensate for multiple appearances of identical terms in the sum.

For the Poincaré case, the MC 1-form (2.4) becomes

$$\Omega = P_a \mathcal{P}^a + \tfrac{1}{2} M_{ab} \mathcal{M}^{ab}, \tag{2.7}$$

while the MC equation (2.3) in components is

$$\mathrm{d}\mathcal{P}^a + \mathcal{P}^c \wedge \mathcal{M}_c{}^a = 0, \qquad \mathrm{d}\mathcal{M}^{ab} + \mathcal{M}^{ac} \wedge \mathcal{M}_c{}^b = 0. \tag{2.8}$$

The first step in the cohomological analysis is to freeze the Lorentz degrees of freedom and construct the most general 2-form that can be built from the translations alone, $\mathcal{P}^a$. The MC equations for these generators ($\mathrm{d}\mathcal{P}^a = 0$) are obtained by putting $\mathcal{M}^{ab} \to 0$ in (2.8). Then, we find that the most general closed invariant 2-form which cannot be written as the differential of an invariant 1-form is

$$\Omega_2 = f_{[ab]}\, \mathcal{P}^a \wedge \mathcal{P}^b \tag{2.9}$$

where $f_{[ab]}$ is a constant second-rank antisymmetric tensor. Therefore, the non-trivial 2-form extensions belong to an antisymmetric tensor representation of the Lorentz group. The 1-form 'potentials' associated with these closed 2-forms are denoted by $\mathcal{Z}^{[ab]}$ and are defined by the equation

$$\mathrm{d}\mathcal{Z}^{[ab]} = \mathcal{P}^a \wedge \mathcal{P}^b. \tag{2.10}$$

From this equation, we obtain the algebra of the corresponding generators, denoted by $Z_{[ab]}$. We find

$$[P_a, P_b] = +\mathrm{i} Z_{[ab]}, \tag{2.11}$$

which implies that there is no central extension of the Poincaré group.

With the rotations included, the extended set of MC 1-forms satisfy the equations

$$\begin{aligned} \mathrm{d}\mathcal{P}^a &= -\mathcal{P}^c \wedge \mathcal{M}_c{}^a, \\ \mathrm{d}\mathcal{M}^{ab} &= -\mathcal{M}^{ac} \wedge \mathcal{M}_c{}^b, \\ \mathrm{d}\mathcal{Z}^{[ab]} &= -\mathcal{Z}^{[ac]} \wedge \mathcal{M}_c{}^b - \mathcal{M}^a{}_c \wedge \mathcal{Z}^{[cb]} + \mathcal{P}^a \wedge \mathcal{P}^b. \end{aligned} \tag{2.12}$$

The associated algebra was introduced before in the literature [9–11]. It is known as the Maxwell algebra.

2.1. *Explicit parametrization*

Here we will introduce explicit parameters labeling the group elements, which will induce an explicit parametrization of all MC 1-forms in terms of the differentials of these parameters. We will first obtain expressions for the MC 1-forms without the Lorentz generators. Specifically, the general element of this coset of the extended group will be parametrized, locally, by

$$g = \mathrm{e}^{\mathrm{i} P_a x^a}\, \mathrm{e}^{\frac{\mathrm{i}}{2} Z_{ab} \theta^{ab}}, \tag{2.13}$$

where $x^a, \theta^{[ab]}$ are the group parameters associated with the generators P_a, Z_{ab}. The component MC 1-forms (2.4) can be computed directly from definition (2.2) and the commutator (2.11) using the Baker–Campbell–Hausdorff formula. The result is

$$\mathcal{P}^a = \mathrm{d}x^a, \qquad \mathcal{Z}^{[ab]} = \mathrm{d}\theta^{[ab]} + \tfrac{1}{2}(x^a\, \mathrm{d}x^b - x^b\, \mathrm{d}x^a). \tag{2.14}$$

If we want to have the explicit expressions of these MC 1-forms when we include the Lorentz degrees of freedom, the right-hand side of all vector and tensor expressions given above must be multiplied by an appropriate orthogonal matrix, U, for each index:

$$\mathcal{P}^a = U^{-1\,a}{}_b\, \mathrm{d}x^b, \qquad \mathcal{Z}^{[ab]} = U^{-1\,a}{}_p\, U^{-1\,b}{}_q\, \left(\mathrm{d}\theta^{[pq]} + \tfrac{1}{2}(x^p\, \mathrm{d}x^q - x^q\, \mathrm{d}x^p)\right). \tag{2.15}$$

U depends on the parameters associated with the Lorentz generators and can be obtained by adding appropriate terms to (2.13). In terms of U, the Lorentz MC 1-forms have the explicit representation $\mathcal{M}^a{}_b = U^{-1a}{}_c\, \mathrm{d}U^c{}_b$.

The vector fields dual to these MC forms (when we freeze the Lorentz degrees of freedom) are

$$Z_{ab} = -\mathrm{i}\frac{\partial}{\partial\theta^{ab}}, \tag{2.16}$$

$$P_a = -\mathrm{i}\left(\frac{\partial}{\partial x^a} + \frac{1}{2}x^r\frac{\partial}{\partial\theta^{ar}}\right), \tag{2.17}$$

$$M_{ab} = -\mathrm{i}\left(x_a\frac{\partial}{\partial x^b} - x_b\frac{\partial}{\partial x^a} + \theta_a{}^r\frac{\partial}{\partial\theta^{br}} - \theta_b{}^r\frac{\partial}{\partial\theta^{ar}}\right), \tag{2.18}$$

and satisfy the commutators

$$\begin{aligned}
&[M_{ab}, M_{cd}] = -\mathrm{i}(\eta_{bc}M_{ad} - \eta_{bd}M_{ac} + \eta_{ad}M_{bc} - \eta_{ac}M_{bd}),\\
&[P_a, M_{bc}] = -\mathrm{i}(\eta_{ab}P_c - \eta_{ac}P_b),\\
&[P_a, P_b] = +\mathrm{i}Z_{ab},\\
&[M_{ab}, Z_{cd}] = -\mathrm{i}(\eta_{bc}Z_{ad} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc} - \eta_{ac}Z_{bd}),\\
&[P_a, Z_{bc}] = 0,\\
&[Z_{ab}, Z_{cd}] = 0.
\end{aligned} \tag{2.19}$$

2.2. *Particle Lagrangian and Noether charges*

To obtain a physical interpretation, one possibility is to construct a particle Lagrangian that is invariant under the extended Poincaré group using the nonlinear realization method [17] for space–time symmetries, see for example [18]. A diffeomorphism-invariant free particle Lagrangian is $\mathcal{L}_0 = m\sqrt{-\dot{x}_a^2}$ and depends on the translation 1-forms only. A possible Lagrangian including also the first extension 1-forms $\mathcal{Z}^{ab}$ (2.14) is

$$\mathcal{L} = m\sqrt{-\dot{x}_a^2} + \tfrac{1}{2}f_{ab}\mathcal{Z}^{ab} = m\sqrt{-\dot{x}_a^2} + \tfrac{1}{2}f_{ab}(\dot{\theta}^{ab} + x^{[a}\dot{x}^{b]}), \tag{2.20}$$

where we have introduced the antisymmetric tensor couplings $f_{ab}(\tau)$ that are considered as new dynamical variables, in addition to the group space coordinates (x^a, θ^{ab}), which are now also functions of the particle proper time. The physical interpretation of the extra variables θ^{ab}, f_{ab} will be given after the equations of motion have been obtained.

This Lagrangian is invariant under translations[8]

$$\delta_P x^a = \epsilon^a, \tag{2.21}$$

$$\delta_P\theta^{ab} = -\tfrac{1}{2}(\epsilon^a x^b - \epsilon^b x^a), \tag{2.22}$$

$$\delta_P f_{ab} = 0, \tag{2.23}$$

and the non-vanishing shifts

$$\delta_Z\theta^{ab} = \epsilon^{ab}, \qquad \delta_Z f_{ab} = 0. \tag{2.24}$$

The Noether charges associated with these symmetries are

$$P_a = p_a - \tfrac{1}{2}p_{ab}x^b, \qquad Z_{ab} = p_{ab}. \tag{2.25}$$

[8] The generators of these transformations are the right invariant vector fields.

If we compute the Poisson bracket among these generators, we find

$$\{P_a, P_b\} = -Z_{ab}, \tag{2.26}$$

i.e. we recover the algebra[9] (2.11).

In the proper time gauge, the Euler–Lagrange equations of motion following from (2.20) are

$$\delta\theta^{ab} \to \dot{f}_{ab} = 0, \tag{2.27}$$

$$\delta f_{ab} \to \dot{\theta}^{ab} + \tfrac{1}{2}(x^a\dot{x}^b - x^b\dot{x}^a) = 0, \tag{2.28}$$

$$\delta x^a \to -m\ddot{x}_a + f_{ab}\dot{x}^b = 0. \tag{2.29}$$

Equation (2.29), with $f_{ab} = e\,F_{ab}$, is the Lorentz force equation determining the motion of a particle of mass m and charge e in an electromagnetic field F_{ab}. Note that for this case, the equation of motion for f_{ab} (2.28) does not affect the dynamics of the coordinates. This equation tells us that $\dot{\theta}^{ab}$ is proportional to the ab component of the angular momentum (or magnetic moment) of the particle[10]. In other words, θ^{ab} is a non-local function of the components of the angular momenta of the particle.

Integration of the equation of motion associated with θ gives $f_{ab} = f^0_{ab} = e\,F^0_{ab}$. We see that this solution spontaneously breaks Lorentz symmetry. If we substitute this solution in the equation of motion for the variable x (2.29), we find that it describes the motion of a particle in a constant, fixed EM field with

$$F_{ab} = F^0_{ab} = \text{constant}. \tag{2.30}$$

It can be obtained from the potential

$$A_a = -\tfrac{1}{2}F^0_{ab}x^b. \tag{2.31}$$

3. Second-level extensions

One can obtain further extensions of the Poincaré group which lead to new generators in higher dimensional representations of the Lorentz group. In order to find them, we apply the same procedure as in the last section, at every level taking as 'translations' all generators of the previous level other than the Lorentz ones, $\mathcal{M}^{ab}$.

For the second extension, we take as 'translations' the 1-forms

$$\mathcal{P}^a, \quad \mathcal{Z}^{[ab]}. \tag{3.1}$$

The calculation results in 20 closed non-trivial 2-forms which can be written as the components of the tensor[11]

$$2\mathcal{P}^a \wedge \mathcal{Z}^{[bc]} - \mathcal{P}^b \wedge \mathcal{Z}^{[ca]} - \mathcal{P}^c \wedge \mathcal{Z}^{[ab]}. \tag{3.2}$$

Again, introducing the second-level potential 1-form $\mathcal{Y}^{a[bc]}$, with the same symmetries as the above 2-form tensor and unfreezing the Lorentz freedom, we find

$$\mathrm{d}\mathcal{Y}^{a[bc]} = -\mathcal{M}^a{}_s \wedge \mathcal{Y}^{s[bc]} - \mathcal{M}^b{}_s \wedge \mathcal{Y}^{a[sc]} - \mathcal{M}^c{}_s \wedge \mathcal{Y}^{a[bs]} + 2\mathcal{P}^a \wedge \mathcal{Z}^{[bc]} - \mathcal{P}^b \wedge \mathcal{Z}^{[ca]} - \mathcal{P}^c \wedge \mathcal{Z}^{[ab]}. \tag{3.3}$$

[9] The reason for the overall sign difference from the starting algebra is that now the generators are *active* operators.

[10] The terminology refers to the space–space components; the space–time components give a Lorentz-boosted momentum (or dipole moment).

[11] This tensor is antisymmetric in $[bc]$ and its totally antisymmetric part vanishes. This leads to four identities, $\epsilon_{abcd}\mathcal{P}^b \wedge \mathcal{Z}^{[cd]} = 0$, leaving $4 \times 6 - 4 = 20$ independent components.

The corresponding generators $Y_{a[bc]}$ appear in the commutators of the original translations with the first-level extensions

$$[P_a, Z_{[bc]}] = 2\mathrm{i}Y_{a[bc]} - \mathrm{i}Y_{b[ca]} - \mathrm{i}Y_{c[ab]}. \tag{3.4}$$

The complete set of the second extension commutators are

$$\begin{aligned}
&[M_{ab}, M_{cd}] = -\mathrm{i}(\eta_{bc}M_{ad} - \eta_{bd}M_{ac} + \eta_{ad}M_{bc} - \eta_{ac}M_{bd}),\\
&[P_a, M_{bc}] = -\mathrm{i}(\eta_{ab}P_c - \eta_{ac}P_b),\\
&[P_a, P_b] = \mathrm{i}Z_{ab},\\
&[M_{ab}, Z_{cd}] = -\mathrm{i}(\eta_{bc}Z_{ad} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc} - \eta_{ac}Z_{bd}),\\
&[P_a, Z_{bc}] = \mathrm{i}(2Y_{abc} - Y_{bca} - Y_{cab}),\\
&[Z_{ab}, Z_{cd}] = 0,\\
&[Y_{pab}, M_{cd}] = -\mathrm{i}(\eta_{bc}Y_{pad} - \eta_{bd}Y_{pac} + \eta_{ad}Y_{pbc} - \eta_{ac}Y_{pbd} + \eta_{pc}Y_{dab} - \eta_{pd}Y_{cab}),\\
&[Y_{pab}, Z_{cd}] = 0,\\
&[Y_{pab}, P_c] = 0,\\
&[Y_{pab}, Y_{qcd}] = 0.
\end{aligned} \tag{3.5}$$

Note that, at this level, the operators Z_{ab} and Y_{abc} generate an Abelian subgroup.

3.1. *Explicit parametrization*

Introducing the second extension parameters $\xi^{a[bc]}$ (coordinates in group space) with the symmetries of $\mathcal{Y}^{a[bc]}$, the coset element in the second extension can be written as

$$g = \mathrm{e}^{\mathrm{i}P_a x^a}\,\mathrm{e}^{\frac{\mathrm{i}}{2}Z_{ab}\theta^{ab}}\,\mathrm{e}^{\frac{\mathrm{i}}{2}Y_{abc}\xi^{abc}}. \tag{3.6}$$

We can then compute, as before, the corresponding MC forms; the ones associated with the translations and first extension are not modified, while the second extension MC forms are found to be

$$\mathcal{Y}^{abc} = \mathrm{d}\xi^{abc} - 2\,\mathrm{d}x^a\theta^{bc} + \mathrm{d}x^b\theta^{ca} + \mathrm{d}x^c\theta^{ab} + \tfrac{1}{2}x^a(x^b\,\mathrm{d}x^c - x^c\,\mathrm{d}x^b). \tag{3.7}$$

The differential operators dual to the 1-forms given above provide a representation of the extended algebra (3.5) (in the summations below, differentiations with respect to variables that are zero—θ^{00}, ξ^{011}, etc—are omitted):

$$Y_{abc} = -\mathrm{i}\frac{\partial}{\partial\xi^{abc}}, \tag{3.8}$$

$$Z_{ab} = -\mathrm{i}\frac{\partial}{\partial\theta^{ab}}, \tag{3.9}$$

$$P_a = -\mathrm{i}\left(\frac{\partial}{\partial x^a} + \frac{1}{2}x^r\frac{\partial}{\partial\theta^{ar}} + \theta^{rs}\frac{\partial}{\partial\xi^{ars}} - \theta^{rs}\frac{\partial}{\partial\xi^{rsa}} - \frac{1}{2}x^r x^s\frac{\partial}{\partial\xi^{rsa}}\right), \tag{3.10}$$

$$\begin{aligned}
M_{ab} = -\mathrm{i}\bigg[&x_a\frac{\partial}{\partial x^b} - x_b\frac{\partial}{\partial x^a} + {\theta_a}^r\frac{\partial}{\partial\theta^{br}} - {\theta_b}^r\frac{\partial}{\partial\theta^{ar}}\\
&+\frac{1}{2}\left({\xi_a}^{rs}\frac{\partial}{\partial\xi^{brs}} - {\xi_b}^{rs}\frac{\partial}{\partial\xi^{ars}}\right) + {\xi^{rs}}_a\frac{\partial}{\partial\xi^{rsb}} - {\xi^{rs}}_b\frac{\partial}{\partial\xi^{rsa}}\bigg].
\end{aligned} \tag{3.11}$$

3.2. Lagrangian associated with the second-level extension

When we include the second extension, the particle Lagrangian becomes

$$\mathcal{L} = m\sqrt{-\dot{x}_a^2} + \tfrac{1}{2} f_{ab}\mathcal{Z}^{ab} + \tfrac{1}{2} f_{abc}\mathcal{Y}^{abc}, \tag{3.12}$$

where the new tensor couplings $f_{abc}(\tau)$ have the symmetries of $\mathcal{Y}^{abc}$ and, together with the new group space coordinates $\xi^{abc}(\tau)$, are considered as new dynamical variables.

Apart from the ordinary Lorentz transformations, the non-trivial transformations leaving the Lagrangian invariant are

$$\delta_P x^a = \epsilon^a, \tag{3.13}$$

$$\delta_P \theta^{ab} = -\tfrac{1}{2}(\epsilon^a x^b - \epsilon^b x^a), \tag{3.14}$$

$$\delta_P \xi^{abc} = -x^a(\epsilon^b x^c - \epsilon^c x^b). \tag{3.15}$$

$$\delta_Z \theta^{ab} = \epsilon^{ab}, \tag{3.16}$$

$$\delta_Z \xi^{abc} = 2x^a\epsilon^{bc} - x^b\epsilon^{ca} - x^c\epsilon^{ab}, \tag{3.17}$$

$$\delta_Y \xi^{abc} = \epsilon^{abc}, \tag{3.18}$$

where $\epsilon^a, \epsilon^{ab}, \epsilon^{abc}$ are arbitrary displacements of the group parameters. The conserved quantities are written as

$$Q = \epsilon^a P_a + \tfrac{1}{2}\epsilon^{ab} Z_{ab} + \tfrac{1}{2}\epsilon^{abc} Y_{abc}, \tag{3.19}$$

from which we obtain the Noether generators

$$\begin{aligned} P_a &= p_a - \tfrac{1}{2} p_{ab} x^b + p_{bca} x^b x^c, \\ Z_{ab} &= p_{ab} + x^c(2p_{cab} - p_{bca} - p_{abc}), \\ Y_{abc} &= p_{abc}. \end{aligned} \tag{3.20}$$

The Poisson brackets among these generators reproduce the algebra (3.5) up to an overall minus sign.

The Euler–Lagrange equations of motion now take the form

$$\delta\xi^{abc} \to \dot{f}_{abc} = 0, \tag{3.21}$$

$$\delta\theta^{ab} \to \dot{f}_{ab} = (-2f_{cab} + f_{abc} + f_{bca})\dot{x}^c, \tag{3.22}$$

$$\delta f_{abc} \to \dot{\xi}^{abc} - 2\dot{x}^a\theta^{bc} + \dot{x}^b\theta^{ca} + \dot{x}^c\theta^{ab} + \tfrac{1}{2}x^a(x^b\dot{x}^c - x^c\dot{x}^b) = 0, \tag{3.23}$$

$$\delta f_{ab} \to \dot{\theta}^{ab} + \tfrac{1}{2}(x^a\dot{x}^b - x^b\dot{x}^a) = 0, \tag{3.24}$$

$$\begin{aligned} \delta x^a \to -m\ddot{x}_a + f_{ab}\dot{x}^b = {} & -\tfrac{1}{2}\dot{f}_{ab}x^b + \tfrac{1}{2}(-2f_{abc} + f_{bca} + f_{cab})\dot{\theta}^{bc} \\ & - \tfrac{1}{2}(-2f_{bca} + f_{cab} + f_{abc})x^b\dot{x}^c. \end{aligned} \tag{3.25}$$

Substituting for $\dot{f}^{ab}, \dot{\theta}^{ab}$ from (3.22), (3.24), we find that the rhs of (3.25) vanishes, so that the equation of motion for x^a depends only on f_{ab}:

$$-m\ddot{x}_a + f_{ab}\dot{x}^b = 0. \tag{3.26}$$

If we integrate equations (3.21) and (3.22) for f_{abc} and f_{ab}, we get

$$f_{abc} = f^0_{abc}, \qquad f_{ab} = \left(-2f^0_{cab} + f^0_{abc} + f^0_{bca}\right)x^c + f^0_{ab}, \qquad f^0_{\dots} = \text{constant} \tag{3.27}$$

Note that these solutions break the extended symmetry spontaneously. If we substitute these expressions back in (3.26), we have

$$-m\ddot{x}_a + f^0_{ab}\dot{x}^b + \left(-2f^0_{cab} + f^0_{abc} + f^0_{bca}\right)x^c\dot{x}^b = 0. \tag{3.28}$$

The resulting equation of motion describes a particle in a given EM field which is linear in the cartesian coordinates and can be obtained from the potential

$$A_a = F^0_{cab}x^b x^c - \tfrac{1}{2}F^0_{ab}x^b, \qquad f^0_{cab} = eF^0_{cab}. \tag{3.29}$$

What makes this possible is the symmetry properties of the quantities f^0_{cab} which imply that the field strength f_{ab} (3.27) satisfies $f_{[ab,c]} = 0$.

Note that, as in the previous level, the equations of motion for f_{ab} and f_{abc} do not affect the dynamics of the coordinates. The variable θ^{ab} retains its old interpretation in terms of the magnetic moment of the particle, while, from (3.23) with $\theta^{ab} = 0$, we see that ξ^{abc} is related to the integral of the first moment of the magnetic moment, i.e. the magnetic quadrupole moment (second moment of the current distribution). Thus, it appears that our physical system is a *distribution* of charged particles, described collectively at this level as the motion of a particle with two sets of moments θ^{ab}, ξ^{abc}, moving in a given EM field. The non-locality of the equations determining the moments suggests that the particles also interact among themselves. This will become apparent at the next level where the equation determining x^a will acquire an extra force term proportional to the magnetic moment.

Writing (3.22) as $\mathrm{d}f_{ab} = (-2f_{cab} + f_{abc} + f_{bca})\,\mathrm{d}x^c$, we can interpret the coefficients f_{abc} as giving the partial derivatives of f_{ab}.

4. Higher level extensions

In this section, we will consider explicitly the higher order extensions up to level 4. Here, as we will see, a new phenomenon appears: we need more than one tensor to describe the new extensions. Moreover, some of the lower level extensions no longer commute with themselves or with the translations.

The procedure can be continued indefinitely. It will become apparent that, at level n, several new tensor extensions of rank $n + 1$ appear.

4.1. Third extension

At the third level, the procedure gives 60 closed 2-forms which can be arranged as the components of two fourth rank tensors with the following symmetries: $\mathcal{S}_1^{(ab)(cd)}, \mathcal{S}_2^{[ab][cd]}$ and the additional antisymmetry $\mathcal{S}_1^{(ab)(cd)} = -\mathcal{S}_1^{(cd)(ab)}$ and $\mathcal{S}_2^{[ab][cd]} = -\mathcal{S}_2^{[cd][ab]}$. These 1-form potentials have, respectively, 45 and 15 independent components and satisfy the equations

$$\begin{aligned}\mathrm{d}\mathcal{S}_1^{abcd} = {}& \mathcal{P}^a\wedge\mathcal{Y}^{cbd} + \mathcal{P}^a\wedge\mathcal{Y}^{dbc} + \mathcal{P}^b\wedge\mathcal{Y}^{cad} + \mathcal{P}^b\wedge\mathcal{Y}^{dac}\\ &- \mathcal{P}^c\wedge\mathcal{Y}^{adb} - \mathcal{P}^c\wedge\mathcal{Y}^{bda} - \mathcal{P}^d\wedge\mathcal{Y}^{acb} - \mathcal{P}^d\wedge\mathcal{Y}^{bca},\end{aligned} \tag{4.1}$$

$$\mathrm{d}\mathcal{S}_2^{abcd} = 4\mathcal{Z}^{ab}\wedge\mathcal{Z}^{cd} + \mathcal{P}^a\wedge\mathcal{Y}^{bcd} - \mathcal{P}^b\wedge\mathcal{Y}^{acd} - \mathcal{P}^c\wedge\mathcal{Y}^{dab} + \mathcal{P}^d\wedge\mathcal{Y}^{cab}. \tag{4.2}$$

In the third extension, the new non-vanishing commutators are those with a total of four free indices. From (4.1), (4.2) it follows that they satisfy

$$[Z_{ab}, Z_{cd}] = 4\mathrm{i}S^2_{abcd}, \tag{4.3}$$

$$[P_a, Y_{bcd}] = \mathrm{i}\left(S^1_{acbd} - S^1_{adbc}\right) + \frac{\mathrm{i}}{3}\left(2S^2_{abcd} - S^2_{acdb} - S^2_{adbc}\right), \tag{4.4}$$

where the new generators S^1_{abcd}, S^2_{abcd} are assumed to have the full symmetries of the corresponding 1-form potentials.

The coset will now be written as

$$g = \mathrm{e}^{\mathrm{i}P_a x^a}\,\mathrm{e}^{\frac{\mathrm{i}}{2}Z_{ab}\theta^{ab}}\,\mathrm{e}^{\frac{\mathrm{i}}{2}Y_{abc}\xi^{abc}}\,\mathrm{e}^{\frac{\mathrm{i}}{8}S^1_{abcd}\sigma_1^{abcd}}\,\mathrm{e}^{\frac{\mathrm{i}}{8}S^2_{abcd}\sigma_2^{abcd}}, \tag{4.5}$$

where $\sigma_1^{abcd}, \sigma_2^{abcd}$ are new scalar parameters having the symmetries of $\mathcal{S}_1^{abcd}, \mathcal{S}_2^{abcd}$. After a long calculation, we obtain the following explicit expressions for the new MC 1-forms:

$$\begin{aligned}\mathcal{S}_1^{abcd} = {}& \mathrm{d}\sigma_1^{abcd} - (\mathrm{d}x^a\xi^{cbd} + \mathrm{d}x^a\xi^{dbc} + \mathrm{d}x^b\xi^{cad} + \mathrm{d}x^b\xi^{dac}) + \mathrm{d}x^c\xi^{adb} + \mathrm{d}x^c\xi^{bda}\\ & + \mathrm{d}x^d\xi^{acb} + \mathrm{d}x^d\xi^{bca} + \tfrac{1}{2}[x^a x^c (x\,\mathrm{d}x)^{bd} + x^b x^d (x\,\mathrm{d}x)^{ac}],\end{aligned} \tag{4.6}$$

$$\begin{aligned}\mathcal{S}_2^{abcd} = {}& \mathrm{d}\sigma_2^{abcd} - (\mathrm{d}x^a\xi^{bcd} - \mathrm{d}x^b\xi^{acd} - \mathrm{d}x^c\xi^{dab} + \mathrm{d}x^d\xi^{cab})\\ & + 2\theta^{ab}(\mathrm{d}\theta^{cd} + (x\,\mathrm{d}x)^{cd}) - 2\theta^{cd}(\mathrm{d}\theta^{ab} + (x\,\mathrm{d}x)^{ab}),\end{aligned} \tag{4.7}$$

where we have used the notation $(x\,\mathrm{d}x)^{ab} = (x^a\,\mathrm{d}x^b - x^b\,\mathrm{d}x^a)$.

4.2. Third-order Lagrangian and equations of motion

With the third-order extensions, the particle Lagrangian becomes

$$\mathcal{L} = m\sqrt{-\dot{x}_a^2} + \tfrac{1}{2}f_{ab}\mathcal{Z}^{ab} + \tfrac{1}{2}f_{abc}\mathcal{Y}^{abc} + \tfrac{1}{8}g_{abcd}\mathcal{S}_1^{abcd} + \tfrac{1}{8}h_{abcd}\mathcal{S}_2^{abcd}, \tag{4.8}$$

where the new tensor couplings $g_{abcd}(\tau),\ h_{abcd}(\tau)$ have the symmetries of $\mathcal{S}_1^{abcd}, \mathcal{S}_2^{abcd}$, respectively, and together with the new group space coordinates $\sigma_1^{abcd}, \sigma_2^{abcd}$ are also treated as new dynamical variables.

This Lagrangian is invariant under the transformations found before plus the following ones for the new variables

$$\delta_P\sigma_1^{abcd} = -\tfrac{3}{2}(\epsilon^a x^b x^c x^d + \epsilon^b x^a x^c x^d - \epsilon^c x^a x^b x^d - \epsilon^d x^a x^b x^c), \tag{4.9}$$

$$\delta_Z\sigma_1^{abcd} = 3(x^a x^d\epsilon^{bc} + x^a x^c\epsilon^{bd} + x^b x^c\epsilon^{ad} + x^b x^d\epsilon^{ac}), \tag{4.10}$$

$$\delta_Y\sigma_1^{abcd} = x^a(\epsilon^{cbd} + \epsilon^{dbc}) + x^b(\epsilon^{cad} + \epsilon^{dac}) - x^c(\epsilon^{adb} + \epsilon^{bda}) - x^d(\epsilon^{acb} + \epsilon^{bca}), \tag{4.11}$$

$$\delta_{S_1}\sigma_1^{abcd} = \epsilon_1^{abcd} \tag{4.12}$$

and

$$\delta_P\sigma_2^{abcd} = (\epsilon^a x^b - \epsilon^b x^a)\theta^{cd} - (\epsilon^c x^d - \epsilon^d x^c)\theta^{ab}, \tag{4.13}$$

$$\delta_Z\sigma_2^{abcd} = \epsilon^{ac}x^b x^d + \epsilon^{bd}x^a x^c - \epsilon^{ad}x^b x^c - \epsilon^{bc}x^a x^d - 2\epsilon^{ab}\theta^{cd} + 2\epsilon^{cd}\theta^{ab}, \tag{4.14}$$

$$\delta_Y\sigma_2^{abcd} = x^a\epsilon^{bcd} - x^b\epsilon^{acd} - x^c\epsilon^{dab} + x^d\epsilon^{cab}, \tag{4.15}$$

$$\delta_{S_2}\sigma_2^{abcd} = \epsilon_2^{abcd}. \tag{4.16}$$

The conserved quantities are written as

$$Q = \epsilon^a P_a + \tfrac{1}{2}\epsilon^{bc}Z_{bc} + \tfrac{1}{2}\epsilon^{abc}Y_{abc} + \tfrac{1}{8}\epsilon_1^{abcd}S^1_{abcd} + \tfrac{1}{8}\epsilon_2^{abcd}S^2_{abcd}, \tag{4.17}$$

from which we obtain the Noether generators

$$P_a = p_a - \tfrac{1}{2}p_{ab}x^b + p_{bca}x^b x^c - \tfrac{3}{4}p^1_{abcd}x^b x^c x^d + \tfrac{1}{2}p^2_{abcd}x^b\theta^{cd} \tag{4.18}$$

$$Z_{bc} = p_{bc} + x^d(2p_{dbc} - p_{bcd} - p_{cdb}) + \tfrac{3}{2}\left(p^1_{abcd} - p^1_{acbd}\right)x^a x^d \\ - \tfrac{1}{2}\left(p^2_{abcd} - p^2_{acbd}\right)x^a x^d - p^2_{bcad}\theta^{ad}, \tag{4.19}$$

$$Y_{abc} = p_{abc} + \tfrac{1}{2}\left(p^1_{abcd} - p^1_{acbd}\right)x^d + \tfrac{1}{3}\left(2p^2_{dabc} - p^2_{dbca} - p^2_{dcab}\right)x^d, \tag{4.20}$$

$$S^1_{abcd} = p^1_{abcd}, \qquad S^2_{abcd} = p^2_{abcd}. \tag{4.21}$$

The Euler–Lagrange equations of motion can be reduced to (3.22), (3.23), (3.24) and the following new equations:

$$\delta\sigma_1{}^{abcd} \to \dot{g}_{abcd} = 0, \tag{4.22}$$

$$\delta\sigma_2{}^{abcd} \to \dot{h}_{abcd} = 0, \tag{4.23}$$

$$\delta g_{abcd} \to \dot{\sigma_1}{}^{abcd} = \dot{x}^a(\xi^{cbd} + \xi^{dbc}) + \dot{x}^b(\xi^{cad} + \xi^{dac}) + \dot{x}^c(\xi^{adb} + \xi^{bda}) \\ + \dot{x}^d(\xi^{acb} + \xi^{bca}) - \frac{1}{2}[x^a x^c(x^b\dot{x}^d - x^d\dot{x}^b) + x^b x^d(x^a\dot{x}^c - x^c\dot{x}^a)], \tag{4.24}$$

$$\delta h_{abcd} \to \dot{\sigma_2}{}^{abcd} = \dot{x}^a\xi^{bcd} - \dot{x}^b\xi^{acd} - \dot{x}^c\xi^{dab} + \dot{x}^d\xi^{cab} \\ - \theta^{ab}(x^c\dot{x}^d - x^d\dot{x}^c) + \theta^{cd}(x^a\dot{x}^b - x^b\dot{x}^a), \tag{4.25}$$

$$\delta\xi^{abc} \to \dot{f}_{abc} = -\dot{x}^d(g_{abcd} - g_{acbd}) - \frac{\dot{x}^d}{3}(2h_{dabc} - h_{dbca} - h_{dcab}), \tag{4.26}$$

$$\delta x^a \to -m\ddot{x}_a + f_{ab}\dot{x}^b = 0, \tag{4.27}$$

where, in reducing (4.25), (4.27), we have used (3.22), (3.23), (3.24). We should remark that, as with (3.25), the rhs of (4.27) is not identically zero, but vanishes because of the other equations of motion. An extra term that vanishes because of (3.24) also appears on the rhs of (3.22).

As in the previous levels, the last terms in (4.25) allow us to relate $\sigma_1{}^{abcd}$ to the third-order moments (octupole) of the current distribution. However, $\dot{\sigma_2}{}^{abcd}$ which vanishes when θ and ξ vanish, must be interpreted differently: it arises from nonlinear couplings of the current with the quadrupole moment ($\dot{x}^a\xi^{cbd}$ terms) as well as of θ with $\dot{\theta}$ (using (3.24); the last two terms in (4.25) are $2\,\theta^{ab}\dot{\theta^{cd}} - 2\theta^{cd}\dot{\theta^{ab}}$).

Integrating (4.22), (4.23), (4.26) and substituting in (3.22), we obtain the equation satisfied by f_{ab}:

$$\dot{f}_{ab} = 3\left(g^0_{cabd} - g^0_{cbad}\right)x^c\dot{x}^d - \left(2h^0_{cdab} - h^0_{cabd} - h^0_{cbda}\right)x^c\dot{x}^d + \left(-2f^0_{cab} + f^0_{abc} + f^0_{bca}\right)\dot{x}^c. \tag{4.28}$$

Writing $2x^c\dot{x}^d = \frac{\mathrm{d}}{\mathrm{d}t}(x^c x^d - 2\theta^{cd})$, as follows from (3.24), we can integrate this equation to get

$$f_{ab} = \tfrac{3}{4}\left(g^0_{cabd} - g^0_{cbad} + g^0_{dabc} - g^0_{dbac}\right)x^c x^d + \tfrac{1}{4}\left(h^0_{cabd} - h^0_{cbad} + h^0_{dabc} - h^0_{dbac}\right)x^c x^d \\ - 2h^0_{abcd}\theta^{cd} + \left(-2f^0_{cab} + f^0_{abc} + f^0_{bca}\right)x^c + f^0_{ab}. \tag{4.29}$$

We observe that, when $h^0_{abcd} \neq 0$, the tensor f_{ab} depends on θ^{ab} in addition to having terms quadratic in the cartesian coordinates. Thus, only part of f_{ab} can be derived from a potential, and the interaction described by f_{ab} can no longer be interpreted as a pure electromagnetic field. The part of f_{ab} that cannot be derived from a potential gives terms to the equation of motion that couple to the magnetic moment. Thus we write (4.27) as

$$m\ddot{x}_a + 2h^0_{abcd}\dot{x}^b\theta^{cd} - h^0_{abcd}x^b\dot{\theta}^{cd} = e\,F_{ab}\dot{x}^b, \tag{4.30}$$

where F_{ab} represents an ordinary external EM field, now quadratic in the coordinates,

$$F_{ab} = \tfrac{3}{4}\big(g^0_{cabd} - g^0_{cbad} + g^0_{dabc} - g^0_{dbac}\big)x^c x^d + \big(-2f^0_{cab} + f^0_{abc} + f^0_{bca}\big)x^c + F^0_{ab}. \tag{4.31}$$

The symmetries of g_{abcd} implies that $F_{[ab,c]} = 0$ and thus can be derived from a potential. The terms depending on h^0_{abcd} in the equation of motion imply a damping effect due to the magnetic moment. As the magnetic moment is determined by the position x, this describes a back-reaction altering the time evolution of x. Thus the dynamics of the motion now depends on the dynamics of the new variables θ^{ab}. We can conjecture that at the next level the tensor f_{ab} will contain terms depending on ξ^{abc}, terms cubic in x^a and terms with the product $\theta^{ab}x^c$, and thus the equation of motion for the coordinates will also couple to the quadrupole moment.

As we did with (3.22), writing (4.26) as $\mathrm{d}f_{abc} = -\mathrm{d}x^d(g_{abcd} - g_{acbd}) - \mathrm{d}x^d(2h_{dabc} - h_{dbca} - h_{dcab})/3$, we can interpret the coefficients g_{abcd} and h_{abcd} as giving the partial derivatives of f_{abc}, which, in turn, determine the partial derivatives of f_{ab}. Thus g_{abcd} and h_{abcd} are, effectively, the second derivatives of f_{ab}. Of course, the physical meaning of these two types of second derivatives is different. The terms with h_{abcd}, leading to non-zero $f_{[ab,c]}$, can be interpreted as magnetic sources. It is known that Maxwell's theory is consistent with the existence of such sources. However, we prefer to interpret these terms as introducing a coupling to the magnetic moment $\dot{\theta}^{ab}$ in the equation of motion for the position variables.

The form (4.30) of the equation of motion, together with the equations determining the evolution of the different moments $(\theta^{ab}, \xi^{abc}, \sigma^{abcd}, \ldots)$, reinforce our conclusion, proposed at the end of section 3, that the physical system described here is a distribution of charges moving consistently (including effects due to non-vanishing moments) in a given EM field. The description is approximated by a series expansion of the field and the corresponding collection of moments of the current distribution, successive levels in the extension procedure giving higher approximations.

At the mathematical level, the coordinates of the extended group space describe the degrees of freedom in the multipole expansion of the current distribution and the induced interactions between them. The symmetry group describes how changes in the coordinates and multipole moments are interrelated in order that a self-consistent interpretation in terms of moving charges in a given external EM field, including back-reaction terms, would be possible.

4.3. Fourth extension

At the fourth level, the procedure gives 204 new extensions which can be grouped as the components of five different fifth rank tensors with definite symmetries. The corresponding 1-form generators, denoted by the symbols $\mathcal{T}_i$, $i = 1, \ldots, 5$, and having the symmetries indicated, satisfy

$$\mathrm{d}\mathcal{T}_1^{(abcd)e} = \mathcal{P}^{(a} \wedge \mathcal{S}_1^{bcd)e}, \tag{4.32}$$

$$\begin{aligned}\mathrm{d}\mathcal{T}_2^{(abc)(de)} = {}& \mathcal{S}_1^{(abc)d} \wedge \mathcal{P}^e + \mathcal{S}_1^{(abc)e} \wedge \mathcal{P}^d - 4(\mathcal{Z}^{d(a} \wedge \mathcal{Y}^{bc)e} + \mathcal{Z}^{e(a} \wedge \mathcal{Y}^{bc)d}) \\ & + \tfrac{4}{3}\big(\mathcal{P}^{(a} \wedge \mathcal{S}_1^{bcd)e} + \mathcal{P}^{(a} \wedge \mathcal{S}_1^{bce)d}\big),\end{aligned} \tag{4.33}$$

$$\begin{aligned}\mathrm{d}\mathcal{T}_3^{(abc)[de]} = {}& (\mathcal{S}_1^{(abc)d} \wedge \mathcal{P}^e - \mathcal{S}_1^{(abc)e} \wedge \mathcal{P}^d + \tfrac{8}{5}(\mathcal{Z}^{d(a} \wedge \mathcal{Y}^{bc)e} - \mathcal{Z}^{e(a} \wedge \mathcal{Y}^{bc)d}) \\ & \times \tfrac{6}{5}\big(\mathcal{S}_2^{adbe} \wedge \mathcal{P}^c - \mathcal{S}_2^{aebd} \wedge \mathcal{P}^c\big)_{(abc)} - \tfrac{4}{5}\big(\mathcal{P}^{(a} \wedge \mathcal{S}_1^{bcd)e} - \mathcal{P}^{(a} \wedge \mathcal{S}_1^{bce)d}\big),\end{aligned} \tag{4.34}$$

$$\mathrm{d}\mathcal{T}_4^{[abc][de]} = \mathcal{S}_2^{[abc]d} \wedge \mathcal{P}^e - \mathcal{S}_2^{[abc]e} \wedge \mathcal{P}^d + 4(\mathcal{Z}^{d[a} \wedge \mathcal{Y}^{bc]e} - \mathcal{Z}^{e[a} \wedge \mathcal{Y}^{bc]d}) + \tfrac{4}{3}\big(\mathcal{P}^{[a} \wedge \mathcal{S}_2^{bcd]e} - \mathcal{P}^{[a} \wedge \mathcal{S}_2^{bce]d} + 2\mathcal{Y}^{e[ab} \wedge \mathcal{Z}^{cd]} - 2\mathcal{Y}^{d[ab} \wedge \mathcal{Z}^{ce]}\big), \qquad (4.35)$$

$$\mathrm{d}\mathcal{T}_5^{[abcd]e} = \mathcal{P}^{[a} \wedge \mathcal{S}_2^{bcd]e} + 2\mathcal{Y}^{e[ab} \wedge \mathcal{Z}^{cd]}, \qquad (4.36)$$

where the subscript (abc) indicates the symmetry operation that must be applied to the expression in parentheses. These fifth rank tensors have, respectively, 84, 60, 36, 20, 4 independent components. We will not investigate further the fourth level extensions.

5. Young tableau symmetries and possible fifth-level extensions

To understand the structure of the higher level extensions, it is helpful to discuss their symmetry properties in terms of Young tableaux (YT). The YT symmetries of all generators (MC 1-forms) up to level 4 are[12]

$$\mathcal{P}^a\ \ydiagram{1},\ \mathcal{Z}^{ab}\ \ydiagram{1,1},\ \mathcal{Y}^{abc}\ \ydiagram{2,1},\ \mathcal{S}_1^{abcd}\ \ydiagram{3,1},\ \mathcal{S}_2^{abcd}\ \ydiagram{2,1,1},$$
$$\mathcal{T}_1^{abcde}\ \ydiagram{4,1},\ \mathcal{T}_2^{abcde}\ \ydiagram{3,2},\ \mathcal{T}_3^{abcde}\ \ydiagram{3,1,1},\ \mathcal{T}_4^{abcde}\ \ydiagram{2,2,1},\ \mathcal{T}_5^{abcde}\ \ydiagram{2,1,1,1}.$$

Note that completely symmetric YT do not appear, as the requirement that the exterior derivative of these 1-forms be given in terms of the wedge product of lower generators implies at least one antisymmetry. The fifth-level generators (MC 1-forms) will have six indices and will transform as the components of sixth-rank tensors with the following possible (in four dimensions) YT symmetries:

$$\mathcal{W}_1^{abcdef}\ \ydiagram{5,1},\ \mathcal{W}_2^{abcdef}\ \ydiagram{4,2},\ \mathcal{W}_3^{abcdef}\ \ydiagram{4,1,1},\ \mathcal{W}_4^{abcdef}\ \ydiagram{3,3},$$
$$\mathcal{W}_5^{abcdef}\ \ydiagram{3,2,1},\ \mathcal{W}_6^{abcdef}\ \ydiagram{3,1,1,1},\ \mathcal{W}_7^{abcdef}\ \ydiagram{2,2,2},\ \mathcal{W}_8^{abcdef}\ \ydiagram{2,2,1,1}.$$

The exterior derivative of each $\mathcal{W}_i$ tensor will then be given in terms of wedge products of pairs of lower order generators whose YT can be combined to give the YT of $\mathcal{W}_i$. For example, the YT $\ydiagram{4,2}$ can be obtained by multiplying the following pairs of YT:

$$\ydiagram{4,1} \otimes \ydiagram{1},\ \ydiagram{3,2} \otimes \ydiagram{1},\ \ydiagram{3,1} \otimes \ydiagram{1,1},\ \ydiagram{2,1} \otimes \ydiagram{2,1}.$$

Thus, we expect $\mathrm{d}\mathcal{W}_2^{(abcd)(ef)}$ to be given as a linear combination of the following four terms: $\mathcal{T}_1^{(abcd)(e} \wedge \mathcal{P}^{f)}$, $\mathcal{P}^{(a} \wedge \mathcal{T}_2^{bcd)(ef)}$, $\mathcal{Z}^{e(a} \wedge \mathcal{S}_1^{bcd)f} + \mathcal{Z}^{f(a} \wedge \mathcal{S}_1^{bcd)e}$ and the $(abcd)(ef)$ part of $\mathcal{Y}^{abe} \wedge \mathcal{Y}^{cdf}$, which is, however, identically zero. Requiring the exterior derivative of this linear combination to be zero, we determine the unknown coefficients (up to an overall constant factor) and thus the equation defining $\mathcal{W}_2$:

$$\mathrm{d}\mathcal{W}_2^{(abcd)(ef)} = 10\mathcal{T}_1^{(abcd)(e} \wedge \mathcal{P}^{f)} - 3\mathcal{P}^{(a} \wedge \mathcal{T}_2^{bcd)(ef)} + 3\big(\mathcal{Z}^{e(a} \wedge \mathcal{S}_1^{bcd)f} + \mathcal{Z}^{f(a} \wedge \mathcal{S}_1^{bcd)e}\big). \qquad (5.1)$$

[12] The third-level extensions have no particular YT symmetry, but can be expressed in terms of such tensors: $\mathcal{S}_1^{(ab)(cd)} = \mathrm{YT}_{31}^{(abc)d} + \mathrm{YT}_{31}^{(abd)c} - \mathrm{YT}_{31}^{(cda)b} - \mathrm{YT}_{31}^{(cdb)a}$, $\mathcal{S}_2^{[ab][cd]} = \mathrm{YT}_{211}^{[abc]d} - \mathrm{YT}_{211}^{[abd]c} - \mathrm{YT}_{211}^{[cda]b} + \mathrm{YT}_{211}^{[cdb]a}$, where the indices on YT tensors indicate the number of boxes in each row of the YT diagram. Similarly, for the fourth extension tensors $\mathcal{T}_i$ defined in subsection 4.3 to have the corresponding YT symmetry, certain symmetry operations must be performed on each $\mathcal{T}_i$.

Not all possible YT symmetries with a given number of indices are present: already at level 3, the symmetry  does not appear. Conversely, some YT symmetries may appear more than once, i.e. the linear combination of possible 2-forms with a given symmetry may not be unique. Due to computer memory limitations, we have done the fifth-level calculations in three dimensions and found that the generator $\mathcal{W}_7$ does not appear, while the generators $\mathcal{W}_3$ and $\mathcal{W}_5$ appear with multiplicities 2 and 3, respectively (the generators $\mathcal{W}_6$ and $\mathcal{W}_8$, being antisymmetric in four indices, cannot exist).

We observe that in the Lagrangian, the coupling to the most symmetric generator (say $k_{(abcd)e}$ corresponding to $\mathcal{T}_1^{(abcd)e}$) will only contribute a term depending on the coordinates to f_{ab}. Thus, restricting attention to these most symmetric extensions, f_{ab} will satisfy $f_{[ab,c]} = 0$,[13] so that an interpretation in terms of an ordinary electromagnetic field will always be possible.

We observe that these coefficients associated with the most symmetric extensions are in one-to-one correspondence with the zero forms used by Vasliliev to describe the 'unfolded dynamics' of the Maxwell equations [19], see also [20].

6. Summary and discussion

We have studied in detail the structure and the particle dynamics of the infinite sequence of extensions of the Poincaré algebra outlined in [1]. We have seen that the generators of the non-central extensions belong to tensor representations of the Lorentz group of increasingly higher rank. We can associate one or more Young tableaux with every extension. Although we have done the calculations in four dimensions, the extensions found also exist in any dimension where their symmetry is allowed. We conjecture that the complete set of extensions constitute an infinite Lie algebra.

We do not have a precise mathematical interpretation for this infinite algebra, but we note that its generator content is organized in levels like the Lorentzian Kac–Moody algebras that are conjectured to be a symmetry of supergravity [15, 16]. It is not completely unnatural that there might be some connection between the two structures, despite the fact that the fields in the Lorentzian Kac–Moody algebras at level zero include the graviton. Following this direction, we have studied analogies with the representations of the over-extension of the G_2, G_2^{++}[14] algebras with respect to A_3.[15] If we disregard the level zero, at level 1 there is a vector corresponding to P_a, at level 2 an antisymmetric two tensor Z_{ab}, at level 3 a mixed generator that corresponds to our Y_{abc}, but at level 4 only one object that corresponds to S^2_{abcd} exists; the generator S^1_{abcd} does not appear. At level 5, only T^4_{abcde}, T^5_{abcde} appear. At level 6, there appear only W^5_{abcdef}, W^8_{abcdef}. Therefore we can conclude that only some of the extensions of the infinite sequence of Poincaré algebras we found appear also at non-zero levels of G_2^{++}. We do not know if one can find an infinite algebra that encompasses all the possible Poincaré extensions.

In order to understand the physical significance of this infinite sequence of extensions of the Poincaré group, we have constructed an invariant Lagrangian that depends linearly on the extensions by introducing tensor coupling 'constants' that we consider as new dynamical variables. The physical system described by this Lagrangian is a distribution of charged particles moving in an external electromagnetic field. The description is approximate: the

[13] Consider the Taylor expansion of f_{ab} about the origin and let $q_{[ab](cde)}$ denote the coefficients of the cubic term. When f_{ab} satisfies $f_{[ab,c]} = 0$, these coefficients must satisfy $q_{[abc](de)} = 0$. These conditions imply that the $q_{[ab](cde)}$ coefficients have the same YT symmetry as that of the coefficients $k_{(abcd)e}$, which satisfy $k_{(abcde)} = 0$, and therefore the $k_{(abcd)e}$ coefficients determine the part cubic in the coordinates of an antisymmetric tensor f_{ab} satisfying $f_{[ab,c]} = 0$.

[14] $d = 5$, $N = 2$ pure supergravity using G_2^{++} was studied in [21].

[15] We have used the computer program SimpLie [22] to study the level structure of the corresponding representations.

particles are described collectively by their multipole moments about the world line of their center of mass and the field by its Taylor expansion about the same line. New terms in the approximation series appear with every extension. The multipoles can be considered as Goldstone bosons. The higher extensions give back-reaction terms describing the effect of the moments on the world line.

We think that sequential extensions of space–time groups including odd generators, using the same methods as in this paper, might be useful in constructing theories containing fermions. We hope to address this point in the future.

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